Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

evidently finite when i is an integer; but for all other values of i, they are infinite, and ultimately divergent, like well known series for the Gamma function. As the convergent parts of the latter function have long been used continuously, so may a limited number of terms of the Riccati series be correctly used for computing the true value of u, when i falls between the integer values.

[Note.—This important inference is further confirmed by the fact that an interpolation formula could certainly be so employed, and that the finite part of the series for w in equation (15) has precisely the form of an interpolation formula, deduced from actual values of u. For example, when $i - 4 = 0$, the series for w in equation (15) reduces to four terms, and the series (16) to five terms, which will determine the value of u, not only when $i = 4$, but also when $i = 3 . 2, 3 . 3, 3 . 7, 3 . 8$, or any value intermediate between 3 and 4. By first augmenting i, as shown by equation (10), the number of terms may be augmented, and u computed to any degree of accuracy.]

* * *

A DISCUSSION OF THE EQUATION OF THE SECOND DEGREE IN TWO VARIABLES.

By Prof. O. H. Mitchell, Marietta, Ohio.

[concluded from page 83.]

ELLIPSES CLASSIFIED.

(α). $R's\Delta < 0$. Since Δ is supposed negative, s is here positive. Distinguishing by the subscripts 1 and 2 the transverse and conjugate values, we have

$$e_1^2 = \frac{2R'}{R' + s}, \quad e_2^2 = \frac{2R'}{R' - s},$$

$$p_1'' = \sqrt{\frac{-4\Delta}{R'(s - R')^2}}, \quad p_2'' = \sqrt{\frac{+4\Delta}{R'(s + R')^2}}.$$

Whence it is seen that e_1 and p_1'' are real, while e_2 and p_2'' are imaginary. Hence, since to the real directrices and foci there corresponds a real value of e, the curve is real. A_1 and A_2 are both real and finite, and (15) shows that the curve has no point at a distance from the centre greater than A_1 or less than A_2. The two values of the semi-parameter are both real. This variety of the ellipse may therefore be described as having *real, finite, unequal axes.*

(β). $R's\Delta > 0$. This implies that s is negative. Hence e_1 and p_2'' are imaginary while e_2 and p_1'' are real. Hence the real value of the ratio e corresponds to the imaginary directrices while the imaginary value of e corresponds to the real directrices. Thus the curve can have no real point, and is called an *imaginary ellipse.* A_1 and A_2, P_1 and P_2, are all imaginary.

(γ). $R's\Delta = 0$. Since $h^2 - ab < 0$, s can not vanish. We have then three cases.

(a). $R' = 0$, $\Delta < 0$. This implies $h = 0$ and $a = b$. Hence $e_1^2 = e_2^2 = 0$, and $p_1'' = p_2'' = \infty$. Thus the four directrices lie at an infinite distance from the centre. From (14) it is seen that the four foci coincide with the centre. $A_1 = A_2 = \sqrt{\frac{-2s\Delta}{s^2a^2}}$. The semi-axes will therefore be real if $s\Delta < 0$, and imaginary if $s\Delta > 0$. Equation (15) becomes

$$x^2 + y^2 = A^2.$$

The curve is a *circle*, real if A^2 is positive ($s\Delta < 0$), imaginary if A^2 is negative ($s\Delta > 0$).

(b). $R' > 0$, $\Delta = 0$. The metrical results (14) all vanish in this case, or say become infinitesimal. Equation (16) becomes

$$(s - R')x^2 + (s + R')y^2 = 0.$$

The curve is, properly speaking, an *ellipse with unequal infinitesimal semi-axes*, since $A_1^2 : A_2^2 = (s - R') : (s + R')$.

(c). $R' = 0$, $\Delta = 0$. This case is the same as the last except that the semi-axes are equal, and $e^2 = 0$. The locus is a *circle with infinitesimal radius.*

HYPERBOLAS CLASSIFIED.

(α). $R's\Delta < 0$. This implies that s is positive. Hence p_1'' is real while p_2'' is imaginary, $1 < e_1^2 < 2$, $e_2^2 < 2$, and $A_1^2 > -A_2^2$, A being real, while A_2 is imaginary. Equation (15) becomes

$$\frac{x^2}{A_1^2} + \frac{y^2}{A_2^2} = 1,$$

where A_2^2 is negative. So that whatever real value y may have from $+\infty$ to $-\infty$, x has a corresponding real value.

(β). $R's\Delta > 0$. Here s is negative, $e_1^2 > 2$, $1 < e_2^2 < 2$, and $A_1^2 < -A_2^2$. Otherwise this curve is the same as the last.

(γ). $R's\Delta = 0$. Since $h^2 - ab > 0$, R' can not vanish. We have three cases.

(a). $s = 0$, $\Delta < 0$. In this case the metrical results (14) are such that each transverse quantity is equal to the corresponding conjugate quantity multiplied by $-\sqrt{-1}$. We have $e_1^2 = e_2^2 = 2$. Since $A_1^2 = -A_2^2$, equation (15) becomes

$$x^2 - y^2 = A^2.$$

This curve is called an *equilateral hyperbola.* The semi-parameters are respectively equal to the semi-axes in analogy with the circle.

(*b*). $s \neq 0$, $\Delta = 0$. Here the metrical results (14) all vanish or, say, become infinitesimal. The foci are at the centre, the directrices pass through the centre (and are all real), and the semi-axes and semi-parameters are infinitesimal. The locus may be considered as an hyperbola with *unequal infinitesimal axes*, for $A_1^2 : A_2^2 = (s - R') : (s + R')$. Strictly speaking neither axis can be distinguished as transverse. Equation (16) becomes

$$(s - R')\,x^2 + (s + R')\,y^2 = 0,$$

which, since $R' > s$, is factorable into two equations of the first degree.

(*c*). $s = 0$, $\Delta = 0$. This case is the same as the last except $A_1^2 = -A_2^2$ and $e_1^2 = e_2^2 = 2$. The locus is an *equilateral hyperbola with infinitesimal axes* whose equation is

$$x^2 - y^2 = 0.$$

PARABOLAS CLASSIFIED.

Since $h^2 = ab$, a and b are of the same sign and $\Delta = -(af^2 - 2fgh + bg^2)$ Hence when Δ is made negative s is positive. We have therefore

$$e_1^2 = 1, \quad e_2^2 = \infty,$$

for every case of the parabola.

(*a*). $R's\Delta < 0$. From (14) we get

$$\begin{aligned} p_1'' &= \infty, \\ p_2'' &= \sqrt{\frac{\Delta}{s^3}}, \\ A_1 &= \infty, \\ A_2 &= \infty, \\ \text{Foci to centre} &= \infty, \\ \text{Focus to transverse directrix} &= \frac{af - gh}{a^{\frac{1}{2}}\,s^{\frac{3}{2}}}, \\ \text{Focus to conjugate directrix} &= \infty, \\ \text{Transverse semi-parameter} &= \frac{af - gh}{a^{\frac{1}{2}}\,s^{\frac{3}{2}}}, \\ \text{Conjugate semi-parameter} &= \infty. \end{aligned}$$

Although A_1 and A_2 are infinite, their ratio $\sqrt{s - R'} : \sqrt{s + R'}$ is infinitesimal. This curve is named a *true parabola.*

The co-ordinates of the centre (12) and (13) are infinite. From (7) we get

$$\theta'\sqrt{b}\,x + \sqrt{a}\,y = \frac{f^2 + g^2 - cs}{\sqrt{af} + \theta'\sqrt{bg} \pm (\sqrt{af} + \theta'\sqrt{bg})},$$

as the equations of the two real directrices of the parabola, of which one is seen to be within a finite distance from the origin, the other infinitely distant. We also get

$$-\theta'\sqrt{a}\,x + \sqrt{b}\,y = \frac{f^2 + g^2}{\theta'\sqrt{ag} - \sqrt{bf} \pm \sqrt{-(\sqrt{af} + \theta'\sqrt{bg})^2}}$$

as the equations of the two imaginary directrices, of which both are at a finite imaginary distance from the origin. These results are obtained from the first form of the value of p, the second being indeterminate.

The co-ordinates of the focus corresponding to that real directrix which is within a finite distance from the origin are

$$x' = \frac{b\,(f^2 - g^2 - cs) + 2fgh}{2s\,(bg - fh)},$$

$$y' = \frac{h\,(f^2 - g^2 + cs) - 2bfg}{2s\,(bg - fh)},$$

and the co-ordinates of the other three foci are infinite. Since both values of p_2 are imaginary, it follows that both values of x_2', y_2' are imaginary though infinite.

From (9) we get as the equations of the axes of the parabola

$$hx + by + \frac{bf + gh}{s} = 0,$$

$$hx - ay + \infty = 0,$$

the first being that of the transverse axis.

Equation (15) is unmanageable for the parabola, the co-efficients being infinite, but (16) does not present this difficulty. It becomes

$$y^2 = \pm\, 2P_1 x.$$

The second member has an ambiguous sign for $\frac{s - R'}{s + R'} = 0$. This means that neither vertex can be considered as the "*left hand vertex*," for each is infinitely distant from the other either to the right or to the left of it. The equation becomes $y^2 = 2P_1 x$ if the origin be taken at that vertex from which the curve proceeds to the right, and $y^2 = -\,2P_1 x$ for the other vertex.

The form conjugate to (16) becomes, after multiplying through by $s - R'$,

$$(s + R')\,x^2 = \pm\sqrt{\frac{-8\Delta}{s - R'}}\,x,$$

an equation which breaks up into the two equations

$$x = 0 \text{ and } (s + R')\,x = \pm\sqrt{\frac{-8\Delta}{s - R'}}.$$

This means that from the point of view of a conjugate vertex a parabola appears as two parallel straight lines infinitely distant from each other.

(β). $R's\Delta > 0$. This is impossible, since s is positive if Δ is negative.

(γ). $R's\Delta = 0$. We have to consider three cases.

(a). $R's = 0$, $\Delta \ddagger 0$. These are impossible conditions, for the first implies $a = b = h = 0$; hence Δ can not be different from zero.

(b). $R's = 0$, $\Delta = 0$. These conditions imply $a = b = h = 0$, and this case has already been noticed as that of a straight line.

(c). $R's \ddagger 0$, $\Delta = 0$. The most elementary way of determining the locus in this case is to put (B) into the form

$$(ax + hy + g)^2 - [(h^2 - ab)y^2 + 2(gh - af)y + (g^2 - ac)] = 0,$$

by the process of completing the square. Then we see that $h^2 - ab = 0$ and $\Delta = 0$ reduce the equation to

$$(ax + hy + g)^2 - (g^2 - ac) = 0, \tag{17}$$

which is seen to be the equation of two parallel lines at a distance from each other equal to

$$2\sqrt{\frac{g^2 - ac}{as}}.$$

These two lines will be real and different 'if $g^2 - ac > 0$,
" " coincident " " $= 0$,
imaginary " " < 0.

In the same way we obtain

$$(by + hx + f)^2 - (f^2 - bc) = 0$$

as the equation of the lines, and

$$2\sqrt{\frac{f^2 - bc}{bs}}$$

as the distance between them. Whence $f^2 - bc$ might replace $g^2 - ac$ in the conditions given above. Hence the conditions in symmetric form are

$$f^2 + g^2 - cs > 0,$$
$$f^2 + g^2 - cs = 0,$$
$$f^2 + g^2 - cs < 0,$$

according as the lines are real and different, coincident, or imaginary.

Applying (9) to find the equations of the axes of this locus we get

$$hx + by + f = 0,$$
$$-hx + ay + \frac{0}{0} = 0;$$

the first being the midway parallel between the lines, the second at right angles to this at an indeterminate distance from the origin. Hence the centre of the locus is found to be indeterminate as also appears from the co-ordinates of the centre.

Since $\Delta = 0$ we get from (7)

$$p_1 = \frac{f^2 + g^2 - cs}{0 \pm 0}$$

$$p_2 = -\frac{f}{bs \pm 0} = -\frac{g}{as \pm 0}.$$

Hence two directrices are coincident at an infinite distance from the origin, the other two coincident at a finite distance. Strictly speaking neither axes can be said to be transverse. But, assuming that s has been made positive, we shall call those results transverse which come from the substitution of the positive value of R. Hence the midway parallel between the lines is the transverse axis and also the conjugate directrix. Hence

$$A_1 = \infty, \quad A_2 = \sqrt{\frac{g_2 - ac}{as}} = \sqrt{\frac{f - bc}{bs}};$$

$$e_1 = 1, \quad e_2 = \infty.$$

An examination of the co-ordinates of the foci shows that they all lie at infinity.

Returning now to the original equations (1) to (6) from which we obtained the values of e, λ, μ, p, x', y', let us inquire how many solutions there ought to be. Dividing each of the first five equations of the set by equation (6), we get rid of the multiplier ρ. We then have, putting $k = x'^2 + y'^2 - e^2p^2$,

$$\begin{aligned} \frac{1 - e^2\lambda^2}{k} &= \frac{a}{c}, \\ \frac{- e^2\lambda\mu}{k} &= \frac{h}{c}, \\ \frac{1 - e^2\mu^2}{k} &= \frac{b}{c}, \\ \frac{e^2p\lambda - x'}{k} &= \frac{g}{c}, \\ \frac{e^2p\mu - y'}{k} &= \frac{f}{c}; \end{aligned} \qquad (18)$$

a system of five equations, each of the fourth degree in the quantities e, λ, μ, p, x', y'. Joining to these the sixth equation $\lambda^2 + \mu^2 = 1$, the system of six equations have $2 . 4^5$ or 2048 solutions. That is, a conic has 2048 each of directrices foci, and eccentricities, when considered from the algebraic point of view.

But our process of elimination yielded only sixteen solutions. We found two values of p, x', y', for each of the two values of e^2, and two values of λ and μ for each pair of values of p. We rejected the negative value of μ as leading to the same geometric results as its positive value. We thus had left eight solutions of the system of equations. The negative value of e may also be rejected as not giving essentially different geometric results. There will be left four solutions. Since the process of elimination consisted only in the addition, subtraction, and multiplication of the equations (division of equations not being employed), it follows that no set of finite solutions could have escaped. The remaining 2032 sets of solutions must therefore be partly or wholly infinite whatever may be the values of a, b, c, f, g, h. That the set of equations (17) may have a set of values of e, λ, μ, p, x', y' partly infinite appears as follows. Suppose a, b, c, f, g, h to have any finite values, then if λ, μ, p be supposed finite, and e, x', y' be supposed infinities of the first order the equations may be satisfied. Furthermore, if e, λ, μ, p be supposed infinites of the first order, and x', y' infinites of the second order, the equations may be satisfied.

THE HOMOGENEOUS EQUATION OF FOUR TERMS TO EXPRESS THE HOMOGRAPHIC DIVISION OF TWO STRAIGHT LINES.

By MR. R. D. BOHANNAN, University of Virginia.

Chasles in his *Traité de Géométrie Supérieure* Art. 137, (Ed. 1852),* says: If two straight lines are divided homographically, and we take upon the one line two fixed points a, c, and upon the other, two fixed points b', d', homographic division will be expressed by the homogeneous equation

$$\frac{am}{cm}\cdot\frac{b'm'}{d'm'} + \lambda\frac{am}{cm} + \mu\frac{b'm'}{d'm'} + \nu = 0.$$

His first demonstration (Art. 137) is indirect, and would not readily suggest itself, and the second demonstration (Art. 140) is cumbrous. The proposition may also be demonstrated in the following manner:—

By Art. 133, homographic division is expressed by the following equation:—

$$am\,.\,b'm' - am\,.\,b'J' - b'm'\,.\,aI + aI\,.\,b'a' = 0,$$

in which I is the point on the first line corresponding to infinity on the second and J' the point on the second corresponding to infinity on the first. Dividing

*Art. 137, Ed. 1852 is 144, Ed. 1880; 133 is 139; 140 has been omitted.